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Ordered Rate Constitutive Theories for Non-Classical Internal Polar Thermoviscoelastic Solids Without Memory

K. S. SURANA¹, F. MOHAMMADI², J. N. REDDY³, AND A. S.

DALKILIC⁴

^{1,2} Mechanical Engineering, University of Kansas, Lawrence, Kansas, USA E-mail: ¹ kssurana@ku.edu
³ Mechanical Engineering, Texas A & M University, College Station, Texas, USA
⁴ Mechanical Engineering, Yildiz Technical University Istanbul, Turkey

Abstract

In a recent paper Surana et. al. [1,2] presented a non-classical internal polar continuum theory for isotropic, homogeneous, incompressible solid matter for small deformation-small strain in which conservation and balance laws were derived by using complete displacement gradient tensor i.e. by incorporating strain tensor and rotation tensor both arising from displacement gradient tensor. It was shown that this theory leads to a more complete thermodynamic framework as it incorporates additional physics due to varying internal rotations between the material points. The currently used classical thermodynamic framework was shown to be a subset of the theory and the thermodynamic framework presented in [1, 2]. In this paper we consider conservation and balance laws derived in [1,2] for non-classical internal polar incompressible solids to present ordered rate constitutive theories for non-classical internal polar thermoviscoelastic solids without memory. We consider isotropic, homogeneous, incompressible matter with small deformation and small strain.

Key Words : Internal polar, Non-classical, Ordered rate constitutive theories, Thermoviscoelastic solids without memory, Conservation and balance laws, Theory of generators and invariants

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Since for such solids the rate of work results in rate of entropy production, the second law of thermodynamics must form the basis for the derivations of the constitutive theories. Conservation and balance laws are used to determine the choice of dependent variables in the constitutive theories. The argument tensors of these dependent variables are chosen using first and second laws of thermodynamics and the physics of internal polar thermoelastic solids without memory. The entropy inequality in terms of Helmholtz free energy density is used to arrive at possible conditions (equalities and inequality) that ensure that when these hold, the entropy inequality is satisfied. These conditions suggest decompositions of the symmetric part of Cauchy stress tensor and symmetric moment tensor into equilibrium and deviatoric parts. The constitutive theories for the equilibrium stress tensor is derived using the conditions resulting form the entropy inequality while the constitutive theories for deviatoric stress tensor is derived using the theory of generators and invariants in conjunction with the conditions resulting from the entropy inequality. It is shown that rate of work due to equilibrium moment tensor is zero, hence the constitutive theory for moment tensor can be derived using deviatoric moment tensor or total moment tensor in conjunction with the theory of generators and invariants. The constitutive theories for the heat vector are derived using the conditions resulting from the entropy inequality as well as using the theory of generators and invariants. The constitutive theories presented here are for the deviatoric stress, moment tensor and heat vector in strain rates of up to orders n and symmetric part of the rotation gradient rate tensors of up to order ${}^{1}n$. The theories presented here describe a broader group of thermoviscoelastic solids without memory in which the conjugate stress, strain, strain rate tensors and conjugate moment, rotation gradient and rotation gradient rate tensors result in energy storage (reversible) as well as dissipation i.e. entropy production which is obviously irreversible. The simplified forms of the constitutive theories are also presented including the first order rate theories.

1. Introduction

The ordered rate constitutive theories presented in this paper are for nonclassical internal polar thermoviscoelastic solids that are isotropic, homogeneous, incompressible and hence small deformation and small strain. The conservation and balance laws for such solid continua were derived in references [1, 2]. For the benefit of the readers the non classical internal polar theory presented in references [1, 2] is summarized here. In a general state of deformation the displacement gradient tensor (${}^{d}\mathbf{J}$) varies between neighboring material points. Its decomposition into symmetric (${}^{d}_{s}\mathbf{J}$) and skew symmetric $\binom{d}{a}\boldsymbol{J}$ tensor separates ${}^{d}\boldsymbol{J}$ into strains and rotations. Thus, varying ${}^{d}\boldsymbol{J}$ between material parts results in varying rotations between them in addition to varying strains. These varying rotations between the material points are obviously due to varying deformation of the continua, hence are internal to the solid matter and are explicitly defined by ${}^{d}_{a}\boldsymbol{J}$ tensor, hence do not require additional external rotational degrees of freedom to describe them.

Just like resistance to deformation, strains and strain rates result in stresses and the conjugate stress and strain, strain rates result in energy storage and dissipation, the varying rotations and their rates between neighboring material points when resisted by the solid matter result in conjugate moment tensor. The moment tensor and its conjugate rotation and rotation rate tensors provide additional mechanism of energy storage and dissipation which is not considered in the current classical continuum theories. The non-classical internal polar continuum theory presented in references [1,2] considers additional physics due to internal varying rotations between neighboring material points and associated conjugate moment tensors in the derivation of conservation and balance laws. This continuum theory is clearly different then micropolar continuum theories published recently in which each material point has six degrees of freedom due to consideration of rotations as external degrees of freedom. We remark that due to consideration of rotations as external degrees of freedom at each material point, the closure to the usual conservation and balance law equations requires additional three equation that are derived for example using conservation of inertia in case of micropolar theories (see references [3, 4]). In the non-classical internal polar theories in references [1,2] that are used in the work presented in this paper rotations are internal, hence explicitly defined by ${}_{a}^{d}\boldsymbol{J}$, thus these theories do not require additional three equations (for example due to conservation of inertia as advocated by Eringen in [3, 4]) to provide closure to the mathematical model.

A comprehensive literature review of micropolar theories, stress couple theories, rotation gradient theories, strain gradient theories with application to bending, vibration, plates and shells were presented in references [1,2]. These are not repeated here. Readers can see references [1,2] for details. It was also demonstrated in references [1,2] that the micropolar theories mentioned above in fact are not the same as those derived in references [1,2]. The constitutive theories for thermoelastic non-classical internal polar solids were presented by Surana et. al. in reference [5]. In reference [5] it was shown that since deformation in thermoelastic solids is reversible, there is no rate of entropy production due to rate of work in such solids, hence entropy inequality in its purest form Clausius-Duhem inequality (a statement of rate of entropy only) has no mechanism for deriving constitutive theories for stress tensor and moment tensor. The constitutive theories for thermoelastic non-classical internal polar solids were derived using strain energy density function [5]. In this paper we consider derivations of constitutive theories for thermoviscoelastic non-classical internal polar solids that have mechanism of energy storage as well as dissipation i.e. in such solids rate of work also results in rate of entropy production. If the deforming volume of matter is in thermodynamic equilibrium, then the constitutive theories must satisfy conservation and balance laws. Thus, the derivations of the constitutive theories under consideration require the following: (i) determination of the dependent variables in the constitutive theories (ii) determination of argument tensors of the dependent variables in the constitutive theories (iii) determination of the mechanism [5] to derive the constitutive theories. Balance of linear momenta, balance of angular momenta and the first law of thermodynamics assume existence of stress tensor, moment tensor and heat vector in the deforming matter without regard to how they are arrived at, hence these balance laws contain no mechanism for deriving constitutive theories for stress tensor, moment tensor and heat vector. Therefore the remaining balance law "second law of thermodynamics (entropy inequality)" must be explored for the derivations of constitutive theories. In the work presented here we consider entropy inequality in conjunction with the theory of generators and invariants [6-25] in the derivations of the constitutive theories.

2. Notations, definitions, measures of rotations, their gradients, stress, strain and moment tensors and preliminary consideration

To distinguish clearly from the reference configuration, current configuration and Lagrangian and Eulerian descriptions we use an over bar to express quantities in the current configuration or the Eulerian description of a quantity. Then $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{A}}$, \bar{V} are length, areas and volumes in the current configuration i.e. deformed where as \boldsymbol{x} , \boldsymbol{A} , V are in the reference configuration (configuration at time $t = t_0 = 0$ i.e. undeformed configuration. As usual $Q(\boldsymbol{x}, t)$ in Lagrangian description of a quantity Q where as $\bar{Q}(\boldsymbol{\bar{x}}, t)$ is Eulerian description of the same quantity Q. Over bar on Q emphasizes the Eulerian description, otherwise without listing its arguments it is not possible to distinguish between Eulerian and Lagrangian descriptions. $\boldsymbol{J}, {}^{d}\boldsymbol{J}, \Phi, \theta, \eta, \boldsymbol{\sigma}, \boldsymbol{m}, \boldsymbol{\Theta}$ define Jacobian of deformation, displacement gradient tensor, Helmholtz free energy density, absolute temperature, entropy density, stress tensor, moment tensor and rotation vector (or tensor of rank one). We note that when the deformation is finite we have

$$\bar{x}_i = \bar{x}_i(x_1, x_2, x_3, t) \tag{2.1}$$

$$x_i = x_i(\bar{x}_1, \bar{x}_2, \bar{x}_3, t) \tag{2.2}$$

If $dx = [dx_1, dx_2, dx_3]^T$ and $d\bar{x} = [d\bar{x}_1, d\bar{x}_2, d\bar{x}_3]^T$ are the components of the length ds and $d\bar{s}$ in the reference and the current configurations and if we neglect the infinitesimals of order two and higher in both configurations, then we obtain

$$\{d\bar{x}\} = [J]\{dx\}$$
(2.3)

$$\{dx\} = [\overline{J}]\{d\overline{x}\} \tag{2.4}$$

with $[J] = [\bar{J}]^{-1}; \ [\bar{J}] = [J]^{-1};$

$$[J][\bar{J}] = [\bar{J}][J] = [I]$$
(2.5)

using Murnaghan's notation

$$[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\bar{x}_1, \bar{x}_2, \bar{x}_3}{x_1, x_2, x_3}\right]; \qquad [\bar{J}] = \left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right] = \left[\frac{x_1, x_2, x_3}{\bar{x}_1, \bar{x}_2, \bar{x}_3}\right] \qquad (2.6)$$

in which the columns of [J] are covariant base vectors $\bar{\boldsymbol{g}}_i$, where as the rows of $[\bar{J}]$ are contravariant base vectors $\bar{\boldsymbol{g}}^i$ [6–9]. [J] and $[\bar{J}]$ are Lagrangian and Eulerian descriptions of the Jacobian of deformation.

When the deformation is small $\bar{x} \simeq x$, hence $[J] \simeq [I]$, with this assumption all stress measures (first and second Piola-Kirchoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for moment tensors. Thus we can write

$$\bar{\boldsymbol{P}} = \boldsymbol{P}; \qquad \bar{\boldsymbol{M}} = \boldsymbol{M}$$
 (2.7)

 $\bar{\boldsymbol{P}}$ and \boldsymbol{P} are average stresses on the oblique plane of the deformed and undeformed tetrahedron and $\bar{\boldsymbol{M}}$ and \boldsymbol{M} are average moment tensors (per unit area) on the same oblique planes. The Cauchy principle for \boldsymbol{P} and \boldsymbol{M} gives

$$\boldsymbol{P} = \boldsymbol{\sigma} \cdot \boldsymbol{n}; \qquad \boldsymbol{M} = \boldsymbol{m} \cdot \boldsymbol{n}$$
 (2.8)

n being unit exterior normal to the oblique plane. $\boldsymbol{\sigma}$ and \boldsymbol{m} are Cauchy stress tensor and Cauchy moment tensor. The displacement gradient matrix $[{}^{d}J]$ and its decomposition into symmetric and antisymmetric parts $[{}^{d}_{s}J]$ and $[{}^{d}_{a}J]$ gives

$${}^{d}J_{ij} = \frac{\partial u_i}{\partial x_j} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \, u_{i,j} \qquad or \qquad [{}^{d}J] = \left[\frac{\partial \{u\}}{\partial \{x\}}\right] \tag{2.9}$$

$$[{}^{d}J] = [{}^{d}_{s}J] + [{}^{d}_{a}J]$$
(2.10)

$$[{}^{d}_{s}J] = \frac{1}{2}([{}^{d}J] + [{}^{d}J]^{T})$$
(2.11)

Let, $\Theta = [\Theta_{x_1}, \Theta_{x_2}, \Theta_{x_3}]^T$ or Θ be the rotations about ox_1 , ox_2 and ox_3 axes of the x-frame, then we can write

$$\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 0 & \Theta_{x_3} & -\Theta_{x_2} \\ -\Theta_{x_3} & 0 & \Theta_{x_1} \\ \Theta_{x_2} & -\Theta_{x_1} & 0 \end{bmatrix}$$
(2.13)

in which

$$\Theta_{x_1} = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right); \quad \Theta_{x_2} = \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right); \quad \Theta_{x_3} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right)$$
(2.14)

Alternatively we can also derive (2.14) as follows

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk} \, \boldsymbol{e}_k \frac{\partial u_j}{\partial x_i} \tag{2.15}$$

$$\nabla \times \boldsymbol{u} = \boldsymbol{e_1} \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \boldsymbol{e_2} \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \boldsymbol{e_3} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$
(2.16)

or

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1(-2\Theta_{x_1}) + \boldsymbol{e}_2(-2\Theta_{x_2}) + \boldsymbol{e}_3(-2\Theta_{x_3})$$
(2.17)

 ϵ_{ijk} is the permutation tensor. The sign difference between (2.14) and (2.17) in due to clockwise or counterclockwise rotations and will only affect the sign of

 \boldsymbol{M} term in the balance of angular momenta. If we use (2.14) as the definition of rotations, then the term containing $\bar{\boldsymbol{M}}$ in the balance of angular momenta must have negative sign. If the rotations in (2.16) are defined as $\Theta_{x_1}, \Theta_{x_2}, \Theta_{x_3}$, then the term containing $\bar{\boldsymbol{M}}$ in the balance of angular momenta must have positive sign. Regardless, the resulting equations and the following derivations are not affected. We note that since

$$[J] = [^{d}J] + [I]$$
(2.18)

$$[_{a}J] = \frac{1}{2}([J] - [J]^{T}) = [_{a}^{d}J]$$
(2.19)

That is the physics and measure of rotations in $[^{d}J]$ and [J] is the same. For small deformation stress tensor $[\varepsilon]$ is defined by

$$[\varepsilon] = \begin{bmatrix} d \\ s \end{bmatrix}$$
(2.20)

We define gradients of rotation $\boldsymbol{\Theta}$ by

$$\begin{bmatrix} \Theta J \end{bmatrix} = \begin{bmatrix} \frac{\partial \{\Theta\}}{\partial \{x\}} \end{bmatrix} \quad or \quad \ \ ^{\Theta}J_{ij} = \frac{\partial \Theta_i}{\partial x_j} \tag{2.21}$$

We also decompose $[{}^{\Theta}J]$ into symmetric and antisymmetric parts $[{}^{\Theta}_{s}J]$ and $[{}^{\Theta}_{a}J]$

$$\begin{bmatrix} \Theta J \end{bmatrix} = \begin{bmatrix} \Theta \\ s \end{bmatrix} + \begin{bmatrix} \Theta \\ a \end{bmatrix}$$
(2.22)

in which

$$\begin{bmatrix} \Theta \\ s \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} \Theta \\ J \end{bmatrix} + \begin{bmatrix} \Theta \\ J \end{bmatrix}^T \right)$$
(2.23)

$$\begin{bmatrix} \Theta \\ a \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} \Theta \\ J \end{bmatrix} - \begin{bmatrix} \Theta \\ J \end{bmatrix}^T \right)$$
(2.24)

Likewise gradients of velocities \boldsymbol{v} and the rotation rates $\boldsymbol{\Theta}$ and their symmetric and antisymmetric parts are given by the following.

$$\left[\frac{\partial\{v\}}{\partial\{x\}}\right] = [L] = [D] + [W] \tag{2.25}$$

$$[D] = \frac{1}{2}([L] + [L]^T); \qquad [W] = \frac{1}{2}([L] - [L]^T) \qquad (2.26)$$

$$\left[\frac{\partial\{\Theta\}}{\partial\{x\}}\right] = \left[^{\Theta}L\right] = \left[^{\Theta}D\right] + \left[^{\Theta}W\right]$$
(2.27)

$$[{}^{\Theta}D] = \frac{1}{2}({}^{\Theta}[L] + [{}^{\Theta}L]^{T}); \qquad [{}^{\Theta}W] = \frac{1}{2}([{}^{\Theta}L] - [{}^{\Theta}L]^{T})$$
(2.28)

3. Conservation and balance laws

Following the derivation of conservation and balance laws for non-classical internal polar solid continua presented in references [1,2], we can write the following for conservation of mass, balance of linear momenta, balance of angular momenta, balance of moments of moments or couples, first law of thermodynamics and the second law of thermodynamics in Lagrangian description (for incompressible case).

$$\rho_0 = |J|\rho(\boldsymbol{x}, t);$$
 $|J| = 1;$ hence
 $\rho_0 = \rho(\boldsymbol{x}, t)$
(3.1)

$$\rho \frac{D \boldsymbol{v}}{D t} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 \qquad (3.2)$$

$$m_{mj,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{3.3}$$

$$\epsilon_{ijk}m_{ij} = 0 \tag{3.4}$$

$$\rho_{0} \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \operatorname{tr}([_{s}\sigma][\dot{\varepsilon}]) - \operatorname{tr}([m][_{s}^{\Theta}\dot{J}]) = 0$$
(3.5)

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} - \operatorname{tr}([_s \sigma][\dot{\varepsilon}]) - \operatorname{tr}([m][_s^{\Theta} \dot{J}]) \le 0$$
(3.6)

in which ρ_0 is the density in the reference configuration, \mathbf{F}^b are body forces, e is specific internal energy, g_i are temperature gradients. We note that stress tensor $\boldsymbol{\sigma}$ has been decomposed into symmetric and antisymmetric tensors i.e.

$$\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma} + {}_{a}\boldsymbol{\sigma} \tag{3.7}$$

and

$$\epsilon_{ijk}\sigma_{ij} = \epsilon_{ijk}({}_a\sigma_{ij}) \qquad as, \quad \epsilon_{ijk}({}_s\sigma_{ij}) = 0$$
(3.8)

$$\operatorname{tr}([{}_{s}\sigma][\dot{\varepsilon}]) = \operatorname{tr}([{}_{s}\sigma][D]) \tag{3.9}$$

$$\operatorname{tr}([m][{}^{\Theta}_{s}J]) = \operatorname{tr}([m][{}^{\Theta}D])$$
(3.10)

A dot in the quantities implies material derivative.

Remarks

(1) From (3.2)-(3.6) we can conclude that *v*, _s*σ*, _a*σ*, *m*, *q*, *e*, Φ, η, θ are dependent variables (twenty five) in the mathematical model (3.2)-(3.6). In this list Φ and η will be eliminated from this list of variables when we consider details of entropy inequality. The specific internal energy *e* for incompressible matter will be a function of θ, hence *e* can be eliminated

from the list of dependent variables. In (3.3) $_{a}\boldsymbol{\sigma}$ depends upon \boldsymbol{m} , hence at this stage $_{a}\boldsymbol{\sigma}$ (three variables) may also be considered in the list of dependent variables. This leaves us with twenty two dependent variables in (3.2)-(3.6) where as we only have seven differential equations ((3.2), (3.3) and (3.5)). Thus we need additional fifteen equations to provide a closure to the model consisting of (3.2), (3.3) and (3.5) in $\boldsymbol{v}, {}_{s}\boldsymbol{\sigma}, {}_{a}\boldsymbol{\sigma}, \boldsymbol{m}, \boldsymbol{q}$ and $\boldsymbol{\theta}$. These additional fifteen equations must come from the constitutive theories for ${}_{s}\boldsymbol{\sigma}, \boldsymbol{m}$ and \boldsymbol{q} .

(2) We note that conservation of inertia as advocated in [3, 4] is not included in (3.1)-(3.6) as additional three equations related to rotations are not required here. The rotation in the present work are due to [J], Jacobian of deformation or alternatively due to $[^{d}J]$, displacement gradient tensor, hence are completely defined by the deformation of solid continua and as a consequence the rotations are not external degrees of freedom at a material point. Thus, with the consideration of rotations at the material points arising and defined from [J] or $[^{d}J]$ as done in the present work, the mathematical model (3.1)-(3.6) requires no additional equations for closure. In the micropolar continuum theories [3, 4] rotations at the material points are additional external three degrees of freedom that are not related to [J] or $[^{d}J]$, hence additional three equations are needed to provide closure to the mathematical model (3.1)-(3.6). These are obtained by considering, for example conservation of inertia, an additional conservation law as advocated by Eringen [3, 4]. Other probabilities may also be worthy of consideration.

4. Dependent variables in the constitutive theories and their argument tensors

The choice of dependent variables in the constitutive theories must be consistent with axiom of casualty [6,7]. The self observable quantities and those that can be derived from them by simple differentiation and/or integration cannot be considered as dependent variables in the constitutive theories. Thus, velocities, temperature, temperature gradient, velocity gradients etc are all ruled out as a choice of dependent variables is the constitutive theories. From the entropy inequality (3.6) we note that ${}_{s}\boldsymbol{\sigma}, \boldsymbol{m}, \Phi, \eta, \boldsymbol{q}$ are possible choices of dependent variables in the constitutive theories. The choice of ${}_{s}\boldsymbol{\sigma}$, \boldsymbol{m} and \boldsymbol{q} as dependent variables in the constitutive theories is also supported by the balance of linear momenta, balance of angular momenta and the energy equation. As mentioned earlier, ${}_{a}\boldsymbol{\sigma}$ are deterministic from (3.3), hence ${}_{a}\boldsymbol{\sigma}$ are not dependent variables in the constitutive theory. At the onset we also consider e, η or Φ, η as dependent variables in the constitutive theories. Choice of e, η or Φ, η is a matter of preference as e, Φ and η are related, hence only two from e, Φ, η need to be considered. We choose Φ and η . Thus, finally we conclude that ${}_{s}\boldsymbol{\sigma}, \boldsymbol{m}, \boldsymbol{q}, \Phi$ and η are possible dependent variables in the constitutive theories. At a later stage in the derivation some of these may be ruled out as dependent variables in the constitutive theories if so warranted by some other considerations.

Next we consider possible choices of argument tensors of the chosen dependent variables keeping in mind the principle of equipresence [6,7] i.e. at the onset all dependent variables in the constitutive theories possibly must contain the same argument tensors. Temperature θ is a natural choice as argument tensor. The choice of \boldsymbol{g} as an argument tensor is necessitated due to dependent variable \boldsymbol{q} in the constitutive theory and the physics of heat conduction. The choice of argument tensors ${}^{d}_{s}\boldsymbol{J}$ and ${}^{\Theta}_{s}\boldsymbol{J}$ or $\boldsymbol{\varepsilon}$ and ${}^{\Theta}\boldsymbol{D}$ is clear due to the fact that ${}^{d}_{s}\dot{\boldsymbol{J}}$ or $\dot{\boldsymbol{\varepsilon}}$ is conjugate ${}_{s}\boldsymbol{\sigma}$ and ${}_{s}^{\Theta}\boldsymbol{\dot{J}}$ or ${}^{\Theta}\boldsymbol{\dot{D}}$ is conjugate to \boldsymbol{m} is rate of work in the energy equation (3.5) and the entropy inequality (3.6). The physics of dissipation in the non-classical internal polar thermoviscoelastic solids necessitates that material derivatives of $\boldsymbol{\varepsilon}$ and ${}^{\Theta}\boldsymbol{D}$ ($\boldsymbol{\varepsilon}_{[1]}$ and ${}^{\Theta}_{s}\boldsymbol{D}_{[1]}$) must also be considered as argument tensor of the dependent variables in the constitutive theories due to presence of dissipation. Just like stress tensor and the conjugate strain rate tensor result in conversion of mechanical work into dissipation or entropy production, rate of rotation gradient tensor and conjugate moment tensor also results in conversion of mechanical work into dissipation or entropy production. Thus ${}^{d}_{s}\dot{J}$ or $\dot{\boldsymbol{\epsilon}}$ and ${}^{\Theta}_{s}\dot{J}$ or ${}^{\Theta}_{s}\dot{\boldsymbol{D}}$ are essential to include as argument tensors of stress tensor and moment tensor respectively for thermoviscoelastic solids. From the entropy inequality $[{}_{s}\sigma]$ and $[\dot{\varepsilon}]$ are conjugate. We note that

$$\operatorname{tr}([{}_{s}\sigma][\dot{\varepsilon}]) = \operatorname{tr}([{}_{s}\sigma][{}_{s}^{d}\dot{J}]) = \operatorname{tr}[{}_{s}\sigma]\left([{}_{s}^{d}\dot{J}] + [{}_{a}^{d}\dot{J}]\right) = \operatorname{tr}([{}_{s}\sigma][\dot{J}]) = ({}_{s}\sigma)_{lk}(\dot{J})_{lk} \quad (4.1)$$

Thus, the entropy inequality (3.6) can be written as

$$\rho_{_{0}}(\dot{\Phi}+\eta\dot{\theta}) + \frac{q_{i}g_{i}}{\theta} - (_{s}\sigma)_{lk}(\dot{J}_{lk}) - (m_{lk})(_{s}^{\Theta}\dot{D})_{lk} \leq 0$$

$$(4.2)$$

In (4.2) $[{}_{s}\sigma]$ and $[\dot{J}]$ are a conjugate pair and we have used, $\operatorname{tr}([m][{}_{s}^{d}\dot{D}]) = (m_{lk})$ $({}_{s}^{\ominus}\dot{D})_{lk}$ for the rate of work due to moment tensor. Jacobian of deformation \boldsymbol{J} , temperature gradient \boldsymbol{g} , temperature θ , ${}_{s}^{\Theta}\boldsymbol{D}$ are obvious choices of the argument tensors of the dependent variables Φ , η , ${}_{s}\boldsymbol{\sigma}$, \boldsymbol{m} and \boldsymbol{q} in the constitutive theories. The material derivative of [J], $\boldsymbol{J}_{[i]}$; $i = 1, 2, \ldots, n$, are fundamental kinematic tensors. Inclusion of $\boldsymbol{J}_{[1]}$ as argument tensor is essential due to dissipative nature of the material behavior. $\boldsymbol{J}_{[i]}$; $i = 2, 3, \ldots, n$, are admissible argument tensors in an ordered theory of up to order n in \boldsymbol{J} as these rate tensors of orders higher than one may also result in dissipation. Likewise inclusion of ${}_{s}^{\Theta}\boldsymbol{D}_{[1]}$ is necessitated due to dissipation associated with \boldsymbol{m} . Rate tensor ${}_{s}^{\Theta}\boldsymbol{D}_{[i]}$; $i = 2, 3, \ldots, {}^{1}n$, should also be included as argument tensors if we consider ordered rate constitutive theory of up to order ${}^{1}n$ as these rate tensors of orders higher than one may also result in dissipation. Using these as argument tensors of Φ , η , ${}_{s}\boldsymbol{\sigma}$, \boldsymbol{m} and \boldsymbol{q} and using principle of equipresence we can write

$$\Phi = \Phi(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$\eta = \eta(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$s\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \overset{\Theta}{s} \boldsymbol{D}, \overset{\Theta}{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

in which $\boldsymbol{J}_{[1]} = \dot{\boldsymbol{J}}$ and ${}_{s}^{\Theta}\boldsymbol{D}_{[1]} = {}_{s}^{\Theta}\dot{\boldsymbol{D}}$ Alternatively we can also write (4.3) using, $\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, ..., n$ instead of $\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, ..., n$.

$$\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

$$\eta = \eta(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

$$s\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

$$\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \varepsilon_{[i]}; i = 1, 2, \dots, n, \ {}^{\Theta}_{s}\boldsymbol{D}, \ {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta)$$

Equation (4.3) or (4.4) are two alternative representations of the argument tensors of the dependent variables in the constitutive theories. We can use either forms in the derivations of the constitutive theories.

5. Entropy inequality, conditions resulting from entropy inequality and final choice of dependent variables and their argument tensors

Using Φ and its argument tensors in (4.3) we can obtain Φ .

$$\dot{\Phi} = \frac{\partial \Phi}{\partial J_{kl}} \dot{J}_{kl} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial (J_{[i]})_{kl}} (\dot{J}_{[i]})_{kl} + \frac{\partial \Phi}{\partial ({}_{s}^{\Theta}D)_{kl}} ({}_{s}^{\Theta}\dot{D})_{kl} + \sum_{j=1}^{n} \frac{\partial \Phi}{\partial ({}_{s}^{\Theta}D_{[j]})_{kl}} ({}_{s}^{\Theta}\dot{D}_{[j]})_{kl} + \frac{\partial \Phi}{\partial g_{i}}\dot{g}_{i} + \frac{\partial \Phi}{\partial \theta}\dot{\theta}$$

$$(5.1)$$

substituting from (5.1) into the entropy inequality (4.2) and collecting terms

$$\left(\rho_{0}\frac{\partial\Phi}{\partial J_{kl}} - {}_{s}\sigma_{lk}\right)\dot{J}_{kl} + \rho_{0}\left(\frac{\partial\Phi}{\partial\theta} + \eta\right)\dot{\theta} + \rho_{0}\sum_{i=1}^{n}\frac{\partial\Phi}{\partial(J_{[i]})_{kl}}(\dot{J}_{[i]})_{kl} + \left(\rho_{0}\frac{\partial\Phi}{\partial({}_{s}^{\Theta}D)_{kl}} - m_{lk}\right)({}_{s}^{\Theta}\dot{D})_{kl} + \rho_{0}\sum_{j=1}^{n}\frac{\partial\Phi}{\partial({}_{s}^{\Theta}D_{[j]})_{kl}}({}_{s}^{\Theta}\dot{D}_{[j]})_{kl} + \rho_{0}\frac{\partial\Phi}{\partial g_{i}}\dot{g}_{i} + \frac{q_{i}g_{i}}{\theta} \leq 0$$

$$(5.2)$$

In order for (5.2) to hold for arbitrary but admissible $\boldsymbol{J}_{[i]}$; i = 1, 2, ..., n, ${}_{s}\boldsymbol{D}_{[j]}^{\theta}$; $j = 1, 2, ..., {}^{1}n$, $\dot{\boldsymbol{g}}$ and $\dot{\boldsymbol{\theta}}$ the following must hold

$$\left(\rho_{0}\frac{\partial\Phi}{\partial J_{kl}} - {}_{s}\sigma_{lk}\right)\dot{J}_{kl} + \left(\rho_{0}\frac{\partial\Phi}{\partial({}_{s}^{\Theta}D)_{kl}} - m_{lk}\right)({}_{s}^{\Theta}\dot{D})_{kl} + \frac{q_{i}g_{i}}{\theta} \le 0 \qquad (5.3)$$

$$\rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) = 0 \tag{5.4}$$

$$\rho_{_{0}}\frac{\partial\Phi}{\partial(J_{[i]})_{kl}} = 0; \quad i = 1, 2, \dots, n$$
(5.5)

$$\rho_0 \frac{\partial \Phi}{\partial ({}^{\Theta}_s D_{[j]})_{kl}} = 0; \quad j = 1, 2, \dots, {}^1 n$$
(5.6)

$$\frac{\partial \Phi}{\partial g_i} = 0; \quad i = 1, 2, 3 \tag{5.7}$$

Remarks

- (1) Equation (5.5) implies that Φ is not a function of $\boldsymbol{J}_{[i]}$; i = 1, 2, ..., n. Equation (5.6) implies that Φ is not a function of ${}_{s}^{\Theta}\boldsymbol{D}_{[j]}$; $j = 1, 2, ..., {}^{1}n$. From (5.7) we can conclude that Φ is not a function of \boldsymbol{g} either. Thus $\Phi = \Phi(\boldsymbol{J}, {}_{s}^{\Theta}\boldsymbol{D}, \theta)$.
- (2) The inequality (5.3) must be maintained in the form it is stated. For example if we set

$$\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - {}_s \sigma_{lk} = 0 \tag{5.8}$$

$$\rho_0 \frac{\partial \Phi}{\partial ({}_s^{\Theta} D)_{kl}} - m_{lk} = 0 \tag{5.9}$$

and

$$\frac{q_i g_i}{\theta} \le 0 \tag{5.10}$$

Then, the entropy inequality is satisfied, however ${}_{s}\boldsymbol{\sigma}$ and \boldsymbol{m} from (5.8) and (5.9) given by

$${}_{s}\sigma_{lk} = \rho_{0} \frac{\partial \Phi}{\partial J_{kl}}$$

$$m_{lk} = \rho_{0} \frac{\partial \Phi}{\partial ({}_{s}^{\Theta}D)_{kl}}$$
(5.11)

are inappropriate as these would imply that ${}_{s}\boldsymbol{\sigma}$ is a function of \boldsymbol{J} and $\boldsymbol{\theta}$ only and \boldsymbol{m} is a function of ${}_{s}^{\Theta}\boldsymbol{D}$ and $\boldsymbol{\theta}$ only which is in contradiction with (4.3). Thus we must maintain (5.3) in the form it is stated.

- (3) From (5.4) we conclude that η is deterministic using Φ , hence η cannot be a dependent variable in the constitutive theories.
- (4) Based on (5.3)-(5.7) and noting that ${}_{s}\boldsymbol{\sigma}, \dot{\boldsymbol{J}}$ and $\boldsymbol{m}, {}_{s}^{\Theta}\dot{\boldsymbol{D}}$ are conjugate pairs we conclude that ${}_{s}\boldsymbol{\sigma}$ cannot be function of ${}_{s}^{\Theta}\dot{\boldsymbol{D}}, {}_{s}^{\Theta}\dot{\boldsymbol{D}}_{[j]}; j = 1, 2, ..., {}^{1}n$ and \boldsymbol{m} cannot be function of $\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, ..., n$, thus we can write the following for the argument tensor of the dependent variables in the constitutive theory.

$$\Phi = \Phi(\boldsymbol{J}, \stackrel{\Theta}{s}\boldsymbol{D}, \theta)$$

$${}_{s}\boldsymbol{\sigma} = {}_{s}\boldsymbol{\sigma}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \boldsymbol{g}, \theta)$$

$$\boldsymbol{m} = \boldsymbol{m}(\stackrel{\Theta}{s}\boldsymbol{D}, \stackrel{\Theta}{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \stackrel{\Theta}{s}\boldsymbol{D}, \stackrel{\Theta}{s}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$
(5.12)

and

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - {}_s \sigma_{lk}\right) \dot{J}_{kl} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_s^{\Theta} D)_{kl}} - m_{lk}\right) ({}_s^{\Theta} \dot{D})_{kl} + \frac{q_i g_i}{\theta} \le 0 \qquad (5.13)$$

We note that in (5.12) the argument tensors of \boldsymbol{q} remain the same as in (4.3) as there are no other considerations to modify these. From (5.13) we do not have any mechanism for deriving constitutive theories for ${}_{s}\boldsymbol{\sigma}$, \boldsymbol{m} and \boldsymbol{q}

5.1. Stress and moment tensor decompositions

To proceed further with (5.13) we consider symmetric stress ${}_{s}\boldsymbol{\sigma}$ and moment \boldsymbol{m} and their decomposition into equilibrium stress ${}_{e}({}_{s}\boldsymbol{\sigma})$ and deviatoric stress tensor ${}_{d}({}_{s}\boldsymbol{\sigma})$ and equilibrium moment tensor ${}_{e}\boldsymbol{m}$ and deviatoric moment tensor ${}_{d}\boldsymbol{m}$.

$${}_{s}\boldsymbol{\sigma} = {}_{e}({}_{s}\boldsymbol{\sigma}) + {}_{d}({}_{s}\boldsymbol{\sigma}) \tag{5.14}$$

$$\boldsymbol{m} = {}_{\boldsymbol{e}}\boldsymbol{m} + {}_{\boldsymbol{d}}\boldsymbol{m} \tag{5.15}$$

Substituting for (5.14) and (5.15) into (5.13) we can write the entropy inequality as

$$\left(\rho_{0}\frac{\partial\Phi}{\partial J_{kl}} - {}_{e}({}_{s}\sigma)_{lk}\right)\dot{J}_{kl} - d_{s} + \left(\rho_{0}\frac{\partial\Phi}{\partial({}_{s}^{\Theta}D)_{kl}} - {}_{e}m_{lk}\right)({}_{s}^{\Theta}\dot{D})_{kl} - d_{m} + \frac{q_{i}g_{i}}{\theta} \le 0 \quad (5.16)$$

where

$$d_{s} = ({}_{d}({}_{s}\sigma)_{lk})(\dot{J}_{kl}) = ({}_{d}({}_{s}\sigma)_{lk})({}_{s}^{d}\dot{J}_{kl}) = ({}_{d}({}_{s}\sigma)_{lk})(\dot{\varepsilon}_{kl})$$

$$d_{m} = ({}_{d}m_{lk})({}_{s}^{\Theta}\dot{D})_{kl}$$
(5.17)

If we let

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - {}_e({}_s\sigma)_{lk} \right) \dot{J}_{kl} = 0$$

$$\left(\rho_0 \frac{\partial \Phi}{\partial ({}_s^{\Theta}D)_{kl}} - {}_em_{lk} \right) ({}_s^{\Theta}\dot{D})_{kl} = 0$$

$$(5.18)$$

Then, the entropy inequality is satisfied if

$$-d_s - d_m + \frac{q_i g_i}{\theta} \le 0 \tag{5.19}$$

For arbitrary but admissible $\dot{\boldsymbol{J}}$ and ${}_{s}^{\Theta}\dot{\boldsymbol{D}}$, (5.19) holds if

$$\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - {}_e({}_s\sigma)_{lk} = 0 \qquad and \qquad \rho_0 \frac{\partial \Phi}{\partial ({}_s^{\Theta}D)_{kl}} - {}_em_{lk} = 0 \tag{5.20}$$

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If (5.20) hold, then the entropy inequality (5.19) is satisfied if, $d_s > 0$, $d_m > 0$, and $\frac{q_i g_i}{\theta} \leq 0$, i.e. rate of work due to $_d({}_s\boldsymbol{\sigma})$ and $_d\boldsymbol{m}$ must be positive and dot product of \boldsymbol{q} and \boldsymbol{g} as ($\theta > 0$) must be negative. Returning back to (5.19) we conclude that $_e({}_s\boldsymbol{\sigma})$ and $_e\boldsymbol{m}$ are deterministic from,

$$_{e}(_{s}\boldsymbol{\sigma})^{T} = \rho_{0}\frac{\partial\Phi}{\partial\boldsymbol{J}} \qquad (_{e}\boldsymbol{m})^{T} = \rho_{0}\frac{\partial\Phi}{\partial_{s}^{\Theta}\boldsymbol{D}}$$
(5.21)

First we note that equilibrium tensors $({}_{e}({}_{s}\boldsymbol{\sigma})$ and ${}_{e}\boldsymbol{m})$ are diagonal tensors with same value of each diagonal component, i.e. ${}_{e}({}_{s}\boldsymbol{\sigma})$ and ${}_{e}\boldsymbol{m}$ have the following forms

$$[_{e}(_{s}\sigma)] = p[I] \tag{5.22}$$

$$[_{e}m] = \widetilde{m}[I] \tag{5.23}$$

p and \widetilde{m} are tensors of rank zero. Using (5.23) and (5.18) we find that

$$\rho_{0} \frac{\partial \Phi}{\partial ({}^{\Theta}_{s}D)_{kl}} ({}^{\Theta}_{s}\dot{D})_{kl} - {}_{e}m_{lk} ({}^{\Theta}_{s}\dot{D})_{kl} = \rho_{0} \frac{\partial \Phi}{\partial ({}^{\Theta}_{s}D)_{kl}} ({}^{\Theta}_{s}\dot{D})_{kl} - \widetilde{m}\,\delta_{lk} ({}^{\Theta}_{s}\dot{D})_{kl} = \rho_{0} \frac{\partial \Phi}{\partial ({}^{\Theta}_{s}D)_{kl}} ({}^{\Theta}_{s}\dot{D})_{kl} - \widetilde{m}\,\mathrm{tr}\,({}^{\Theta}_{s}\dot{D}) = 0$$

$$(5.24)$$

But tr $\binom{\Theta}{s}\dot{D} = 0$, hence from (5.24) we obtain

$$\rho_0 \frac{\partial \Phi}{\partial ({}^{\Theta}_s D)_{kl}} ({}^{\Theta}_s \dot{D})_{kl} = 0$$
(5.25)

Equation (5.25) obviously implies that

$$\frac{\partial \Phi}{\partial ({}^{\Theta}_{s}D)_{kl}} = 0 \tag{5.26}$$

Hence Φ is not a function of ${}_{s}^{\Theta}\boldsymbol{D}$ and ${}_{e}\boldsymbol{m}$ is of no consequence as it produces zero rate of work. Thus, now we have the following.

$$\Phi = \Phi \left(\boldsymbol{J}, \theta \right)$$

$$d(_{s}\boldsymbol{\sigma}) = d(_{s}\boldsymbol{\sigma}) \left(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, \boldsymbol{g}, \theta \right)$$

$$d\boldsymbol{m} = _{d}\boldsymbol{m} \left({}_{s}^{\Theta}\boldsymbol{D}, {}_{s}^{\Theta}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta \right)$$

$$\boldsymbol{q} = \boldsymbol{q} \left(\boldsymbol{J}, \boldsymbol{J}_{[i]}; i = 1, 2, \dots, n, {}_{s}^{\Theta}\boldsymbol{D}, {}_{s}^{\Theta}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta \right)$$

$$-d_{s} - d_{m} + \frac{q_{i}g_{i}}{\theta} \leq 0$$
(5.27)

$$d_s > 0, \quad d_m > 0, \quad and \qquad \frac{q_i g_i}{\theta} \le 0$$
 (5.29)

$$_{e}(_{s}\boldsymbol{\sigma}) = \rho_{_{0}}\frac{\partial\Phi}{\partial\boldsymbol{J}} \tag{5.30}$$

In (5.28), d_s and d_m are defined by (5.17). Constitutive theory for $_{e}(_{s}\boldsymbol{\sigma})$ is deterministic from (5.30). We recall that $_{d}(_{s}\boldsymbol{\sigma})$ and \boldsymbol{q} in (5.27) can also be written as

$$d({}_{s}\boldsymbol{\sigma}) = d({}_{s}\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \dots, n, \, \boldsymbol{g}, \theta)$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}, \, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \dots, n, \, {}_{\boldsymbol{s}}^{\boldsymbol{\Theta}}\boldsymbol{D}, \, {}_{s}^{\boldsymbol{\Theta}}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \, \boldsymbol{g}, \theta)$$
(5.31)

This is due to the fact that

$$\operatorname{tr}\left(({}_{s}\sigma)(\dot{J})\right) = \operatorname{tr}\left(({}_{s}\sigma)(\dot{\varepsilon})\right)$$
(5.32)

6. Derivation of constitutive theories for $_{e}(_{s}\boldsymbol{\sigma}), _{d}(_{s}\boldsymbol{\sigma}), _{d}\boldsymbol{m}$ and \boldsymbol{q}

In this section we consider the details of the derivations of the constitutive theories for $_{e}(_{s}\boldsymbol{\sigma}), _{d}(_{s}\boldsymbol{\sigma}), _{d}\boldsymbol{m}$ and \boldsymbol{q} .

6.1. Constitutive theory for $_{e}(_{s}\boldsymbol{\sigma})$

For small deformation (infinitesimal) small strain, the solid matter is incompressible during deformation, hence $\rho(x,t) = \rho_0$, which implies that |J| = 1, hence $\Phi = \Phi(\theta)$, and $\frac{\partial \Phi}{\partial J} = 0$. Therefore the constitutive for $_e(_s \sigma)$ cannot be derived using (5.30). Instead the incompressibility condition |J| = 1 must be enforced.

For incompressible matter

$$tr[\bar{D}] = tr[\bar{L}] = tr[\dot{J}J^{-1}] = \dot{J}_{kl}(J^{-1})_{lk} = \dot{J}_{kl}\delta_{lk}$$
(6.1)

If (6.1) holds, then

$$p\dot{J}_{kl}\delta_{lk} = 0 = p(\theta)\dot{J}_{kl}\delta_{lk} = 0$$
(6.2)

must hold. $p(\theta)$ is an arbitrary Lagrange multiplier. Using first equation of (5.18) and adding (6.2) (since it is zero) we obtain

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - {}_e({}_s\sigma)_{lk} + p(\theta)\delta_{lk}\right)\dot{J}_{kl} = 0$$
(6.3)

using $\frac{\partial \Phi}{\partial J} = 0$ in (6.3), we obtain

$$\left(-{}_{e}({}_{s}\sigma)_{lk}+p(\theta)\delta_{lk}\right)\dot{J}_{kl}=0$$
(6.4)

For (6.4) to hold for an arbitrary but admissible \dot{J} , we must have

$$-_{e}(_{s}\sigma)_{lk} + p(\theta)\delta_{lk} = 0 \tag{6.5}$$

or

$$_{e}(_{s}\boldsymbol{\sigma}) = p(\theta)\boldsymbol{I} \tag{6.6}$$

In (6.6), $p(\theta)$ is called mechanical pressure. Clearly $p(\theta)$ is not deterministic from deformation. Mechanical pressure is also called mean normal stress. If we consider compressive pressure to be positive then $p(\theta)$ in (6.6) can be replaced with $-p(\theta)$

6.2. Ordered rate constitutive theories of up to order n for $_d(_s\boldsymbol{\sigma})$

We consider conjugate pairs ${}_{s}\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}$, then we have (using $\boldsymbol{\varepsilon}_{[0]}$ for $\boldsymbol{\varepsilon}$).

$${}_{s}\boldsymbol{\sigma} = p(\theta)\boldsymbol{I} + {}_{d}({}_{s}\boldsymbol{\sigma})(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \dots, n, \boldsymbol{g}, \theta)$$
(6.7)

We consider the most general form of the constitutive theory for $_{d(s\boldsymbol{\sigma})}$ based on its argument tensors defined in (6.7). We use theory of generators and invariants in doing so [6–25]. In this theory we express $_{d(s\boldsymbol{\sigma})}$ as a linear combination of the combined generators of the argument tensors of $_{d(s\boldsymbol{\sigma})}$ in (6.7). The coefficients in this linear combination are functions of the combined invariants of the same argument tensors of $_{d(s\boldsymbol{\sigma})}$ and temperature θ in (6.7). The material coefficients are derived by considering Taylor series expansion of each coefficient in the linear combination in the combined invariants and temperature θ about a known configuration $\underline{\Omega}$. $_{d(s\boldsymbol{\sigma})}$ is a symmetric tensor of rank two. The argument tensors $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \ldots, n$ are also symmetric tensor of rank two, but \boldsymbol{g} and θ are tensors of rank one and zero. Let ${}^{\boldsymbol{\sigma}}\boldsymbol{\zeta}^{i}$, $i = 1, 2, \ldots, N$ be the combined generators [6–25] of the argument tensors of $_{d(s\boldsymbol{\sigma})}$ in (6.7) that are symmetric tensors of rank two and let ${}^{\boldsymbol{\sigma}}\boldsymbol{L}^{j}$, $j = 1, 2, \ldots, M$ be the combined invariants of the argument tensors of $_{d(s\boldsymbol{\sigma})}$ in (6.7). Then, we can express $_{d(s\boldsymbol{\sigma})}$ in the current configuration using

$$[_{d}(_{s}\sigma)] = {}^{\sigma}\!\underline{\alpha}^{i}[I] + \sum_{i=1}^{N} {}^{\sigma}\!\underline{\alpha}^{i}[{}^{\sigma}\!\underline{G}^{i}]$$

$$(6.8)$$

in which

$${}^{\sigma}\underline{\alpha}^{i} = {}^{\sigma}\underline{\alpha}^{i}({}^{\sigma}\underline{I}^{j}; j = 1, 2, \dots, M, \theta); i = 0, 1, \dots, N$$

$$(6.9)$$

To determine material coefficient from ${}^{\sigma} \underline{\alpha}^i$; i = 0, 1, ..., N in (6.8), we consider Taylor series expansion of the coefficients ${}^{\sigma} \underline{\alpha}^i$; i = 0, 1, ..., N about a known configuration $\underline{\Omega}$ in ${}^{\sigma} \underline{I}^j$, j = 1, 2, ..., M and θ and retain only up to linear terms in the invariants (for simplicity) and the temperature θ .

$${}^{\sigma}\underline{\alpha}^{i} = {}^{\sigma}\underline{\alpha}^{i}\Big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\underline{\alpha}^{j})}{\partial({}^{\sigma}\underline{I}^{j})}\Big|_{\underline{\Omega}} \Big({}^{\sigma}\underline{I}^{j} - ({}^{\sigma}\underline{I}^{j})_{\underline{\Omega}}\Big) + \frac{\partial({}^{\sigma}\underline{\alpha}^{i})}{\partial\theta}\Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}), \quad i = 0, 1, \dots, N$$

$$(6.10)$$

When (6.10) is substituted in (6.8), we obtain final form of the most general constitutive theory for $_{d}(_{s}\boldsymbol{\sigma})$ of order n. Details of the material coefficients are given in the following. Substituting (6.10) in (6.8).

$$\begin{aligned} [_{d}(_{s}\sigma)] &= \left({}^{\sigma}\underline{\alpha}^{0}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\underline{\alpha}^{0})}{\partial({}^{\sigma}\underline{I}^{j})}\big|_{\underline{\Omega}} \left({}^{\sigma}\underline{I}^{j} - ({}^{\sigma}\underline{I}^{j})_{\underline{\Omega}}\right) + \frac{\partial({}^{\sigma}\underline{\alpha}^{0})}{\partial\theta}\Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) [I] \\ &+ \sum_{i=1}^{N} \left({}^{\sigma}\underline{\alpha}^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\underline{\alpha}^{i})}{\partial({}^{\sigma}\underline{I}^{j})}\Big|_{\underline{\Omega}} ({}^{\sigma}\underline{I}^{j} - ({}^{\sigma}\underline{I}^{j})_{\underline{\Omega}}) + \frac{\partial({}^{\sigma}\underline{\alpha}^{i})}{\partial\theta}\Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) [{}^{\sigma}\underline{G}^{i}] \end{aligned}$$
(6.11)

Collecting coefficients (those defined in the known configuration $\underline{\Omega}$) of [I], $({}^{\sigma}\underline{I}^{j})[I]$, $[{}^{\sigma}\underline{G}^{i}]$, $({}^{\sigma}\underline{I}^{j})[{}^{\sigma}\underline{G}^{i}]$, $(\theta - \theta_{\underline{\alpha}})[{}^{\sigma}\underline{G}^{i}]$ and $(\theta - \theta_{\underline{\alpha}})[I]$ in (6.11) and defining

$$\begin{split} \underline{\sigma}^{0} \Big|_{\underline{\alpha}} &= {}^{\sigma} \underline{\alpha}^{0} \Big|_{\underline{\alpha}} + \sum_{j=1}^{M} \frac{\partial ({}^{\sigma} \underline{\alpha}^{0})}{\partial ({}^{\sigma} \underline{I}^{j})} \Big|_{\underline{\alpha}} ({}^{\sigma} \underline{I}^{j})_{\underline{\alpha}}; \\ {}^{\sigma} \underline{a}_{j} &= \frac{\partial ({}^{\sigma} \underline{\alpha}^{0})}{\partial ({}^{\sigma} \underline{I}^{j})} \Big|_{\underline{\alpha}}, \ j = 1, 2, \dots, M \\ {}^{\sigma} \underline{b}_{i} &= {}^{\sigma} \underline{\alpha}^{i} \Big|_{\underline{\alpha}} + \sum_{j=1}^{M} \frac{\partial ({}^{\sigma} \underline{\alpha}^{i})}{\partial ({}^{\sigma} \underline{I}^{j})} \Big|_{\underline{\alpha}}; \ i = 1, 2, \dots, N \\ {}^{\sigma} \underline{c}_{ij} &= \frac{\partial ({}^{\sigma} \underline{\alpha}^{i})}{\partial ({}^{\sigma} \underline{I}^{j})} \Big|_{\underline{\alpha}} \quad i = 1, 2, \dots, M \\ {}^{\sigma} \underline{d}_{i} &= -\frac{\partial ({}^{\sigma} \underline{\alpha}^{i})}{\partial \theta} \Big|_{\underline{\alpha}}, \ i = 1, 2, \dots, N \\ ({}^{\sigma} \underline{\alpha}_{tm}) &= -\frac{\partial ({}^{\sigma} \underline{\alpha}^{0})}{\partial \theta} \Big|_{\underline{\alpha}} \end{split}$$

$$(6.12)$$

Equation (6.11) can be written as

$$\begin{aligned} [{}_{d}({}_{s}\sigma)] &= \underline{\sigma}^{0} \big|_{\underline{\Omega}} [I] + \sum_{j=1}^{M} {}^{\sigma}\underline{a}_{j} {}^{\sigma}\underline{I}^{j} [I] + \sum_{i=1}^{N} {}^{\sigma}\underline{b}_{i} [{}^{\sigma}\underline{G}^{i}] + \sum_{i=1}^{N} \sum_{j=1}^{M} {}^{\sigma}\underline{c}_{ij} {}^{\sigma}\underline{I}^{j} [{}^{\sigma}\underline{G}^{i}] \\ &+ \sum_{i=1}^{N} {}^{\sigma}\underline{d}_{i} (\theta - \theta_{\underline{\Omega}}) [{}^{\sigma}\underline{G}^{i}] - {}^{\sigma}\underline{\alpha}_{tm} (\theta - \theta_{\underline{\Omega}}) [I] \end{aligned}$$
(6.13)

 $\underline{\sigma}^{0}|_{\underline{\Omega}}$ is the initial stress field in the configuration $\underline{\Omega}$. This constitutive theory for $_{d}(_{s}\boldsymbol{\sigma})$ requires (M + N + MN + N + 1) material coefficients. The material coefficients defined in (6.12) are functions of $\theta_{\underline{\Omega}}$ and $(^{\sigma}\underline{I}^{j}); j = 1, 2, \ldots, M$ in the known configuration $\underline{\Omega}$. The constitutive theory is based on integrity (complete basis), hence is complete. Simplified constitutive theories for $_{d}(_{s}\boldsymbol{\sigma})$ can be derived using (6.13) by only retaining the generators that are of interest. We consider some in the following.

6.2.1. Simplified linear rate constitutive theories of up to order n for $d(s\sigma)$

Using the general ordered rate constitutive theory of up to order n in (6.13) we consider simplified rate constitutive theories of up to order n based on the following assumptions.

- (1) The constitutive theories are linear in the components of each argument tensors of $_{d}(_{s}\boldsymbol{\sigma})$.
- (2) We neglect all terms containing the products of the components of the argument tensors. i.e the terms containing the products of the generators and the invariants as well as the terms containing products of $(\theta \theta_{\underline{\alpha}})$ with generators and invariants.
- (3) We assume the constitutive theories for $_d(_s\boldsymbol{\sigma})$ are independent of \boldsymbol{g} .

Based on these assumptions, the constitutive theory for $_{d}(_{s}\boldsymbol{\sigma})$ will only contain tensors $[\boldsymbol{\varepsilon}_{[i]}]$; i = 0, 1, ..., n as generators and their traces as invariants, $\underline{\sigma}^{0}|_{\underline{\alpha}}$ and $^{\sigma}\underline{\alpha}_{tm}(\theta - \theta_{\underline{\alpha}})[I]$ and we can write the following for $_{d}(_{s}\boldsymbol{\sigma})$.

$$[_{d}(_{s}\sigma)] = \underline{\sigma}^{0}|_{\underline{\alpha}}[I] + a_{1}[\varepsilon_{[0]}] + a_{2}(\operatorname{tr}([\varepsilon_{[0]}])[I] + \sum_{i=1}^{n} b_{i}^{1}[\varepsilon_{[i]}] + \sum_{i=1}^{n} b_{i}^{2}(\operatorname{tr}([\varepsilon_{[i]}])[I] - (^{\sigma}\underline{\alpha}_{tm})(\theta - \theta_{\underline{\alpha}})[I]$$

$$(6.14)$$

The material coefficients a_1 , a_2 , b_i^1 , b_i^2 are functions of the invariants and temperature θ in configuration $\underline{\Omega}$. We can write (6.14) in the following form

using Voigt's notation (in the absence of the first and the last terms in (6.14)).

$$\{_d(s\sigma)\} = [\underline{a}]\{\varepsilon_{[0]}\} + \sum_{i=1}^n [\underline{b}_i]\{\varepsilon_{[i]}\}$$
(6.15)

in which

$$\{_{d}(_{s}\sigma)\}^{T} = [_{d}(_{s}\sigma)_{x_{1}x_{1}}, \ _{d}(_{s}\sigma)_{x_{2}x_{2}}, \ _{d}(_{s}\sigma)_{x_{3}x_{3}}, \ _{d}(_{s}\sigma)_{x_{2}x_{3}}, \ _{d}(_{s}\sigma)_{x_{3}x_{1}}, \ _{d}(_{s}\sigma)_{x_{1}x_{2}}]$$

$$(6.16)$$

and

$$[\underline{a}] = \begin{bmatrix} a_1 + a_2 & a_2 & a_2 & 0 & 0 & 0 \\ a_2 & a_1 + a_2 & a_2 & 0 & 0 & 0 \\ a_2 & a_2 & a_1 + a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix}$$
(6.17)

The coefficients of $[\underline{b}_i]$ can be obtained by using (6.17) and replacing a_1 , a_2 with b_i^1 and b_i^2 . In (6.15) $_d({}_s\boldsymbol{\sigma})$ is a linear function of the tensors $\boldsymbol{\varepsilon}_{[i]}$; $i = 0, 1, \ldots, n$, hence the theories defined by (6.15) are simplified linear rate theories of orders $1, 2, \ldots, n$. We note that at the very least $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ are essential for elasticity and dissipation. This simplified theory of order n is an illustration of the many possible linear and nonlinear constitutive theories are possible from the general form (6.13).

6.3. Ordered rate constitutive theories up to orders ${}^{1}n$ for ${}_{d}\boldsymbol{m}$ or \boldsymbol{m} In view of the fact that ${}_{e}\boldsymbol{m}$ results in zero rate of work, when deriving constitutive theory for the moment tensor we can either consider ${}_{d}\boldsymbol{m}$ or \boldsymbol{m} as $\boldsymbol{m} = {}_{d}\boldsymbol{m} + {}_{e}\boldsymbol{m}$, but rate of work due to ${}_{e}\boldsymbol{m}$ is zero, thus we consider ${}_{d}\boldsymbol{m}$ and its argument tensors in (5.27).

$$_{d}\boldsymbol{m} = {}_{d}\boldsymbol{m} ({}_{s}^{\Theta}\boldsymbol{D}, {}_{s}^{\Theta}\boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \boldsymbol{g}, \theta)$$

$$(6.18)$$

 $_{d}\boldsymbol{m}$ in (6.18) can be replaced by \boldsymbol{m} and we have

$$\boldsymbol{m} = \boldsymbol{m} \begin{pmatrix} \boldsymbol{\Theta} \boldsymbol{D}, \ \boldsymbol{s} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \ \boldsymbol{g}, \theta \end{pmatrix}$$
(6.19)

We can use either of (6.18) and (6.19) in the derivation of the constitutive theory for moment tensor. We consider most general form constitutive theory for the moment tensor \boldsymbol{m} using its argument tensor in (6.19) and temperature θ . We use theory of generators and invariants [6–25] in the derivative of the constitutive theory. In the theory we express [m] as a linear combination of the combined generators of the argument tensors of \boldsymbol{m} in (6.19). The coefficients in the linear combination are functions of the combined invariants of the same argument tensors of [m] in (6.19). The material coefficients are derived by considering Taylor series expansion of each coefficient in the linear combination in the combined invariants and temperature θ about a known configuration $\underline{\Omega}$. [m] is a symmetric tensor of rank two. The argument tensors $[{}^{\Theta}_{s}D], [{}^{\Theta}_{s}D_{[j]}]; j = 1, 2, \ldots, {}^{1}n$ are symmetric tensors of rank two, but \boldsymbol{g} and $\boldsymbol{\theta}$ are tensors of rank one and zero respectively. Let, $[{}^{m}\underline{G}^{i}]; i = 1, 2, \ldots, N$ be the combined generators of the argument tensors of \boldsymbol{m} that are symmetric tensor of rank two and ${}^{m}\underline{I}^{j}; j = 1, 2, \ldots, M$ be the combined invariants of the same argument tensors. Then we can express \boldsymbol{m} in the current configuration using

$$[m] = {}^{m}\underline{\alpha}^{i}[I] + \sum_{i=1}^{N} {}^{m}\underline{\alpha}^{i}[{}^{m}\underline{G}^{i}]$$
(6.20)

The coefficients ${}^{m}\underline{\alpha}^{i}$; $i = 0, 1, ..., \underline{N}$ are functions of θ and ${}^{m}\underline{I}^{j}$; $j = 1, 2, ..., \underline{M}$ in the current configuration i.e.

$${}^{m}\underline{\alpha}^{i} = {}^{m}\underline{\alpha}^{i}(\theta, {}^{m}\underline{I}^{j}, j = 1, 2, \dots, \underline{M},); i = 0, 1, \dots, \underline{N}$$
(6.21)

To determine material coefficient from (6.21), we consider Taylor series expansion of each ${}^{m} \alpha^{i}$; i = 0, 1, ..., N in θ and ${}^{m} L^{j}, j = 1, 2, ..., M$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in θ and invariants (for simplicity)

$${}^{m}\underline{\alpha}^{i} = {}^{m}\underline{\alpha}^{i} \Big|_{\underline{\Omega}} + \sum_{j=1}^{\underline{M}} \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial({}^{m}\underline{I}^{j})} \Big|_{\underline{\Omega}} \Big({}^{m}\underline{I}^{j} - ({}^{m}\underline{I}^{j})_{\underline{\Omega}}\Big) + \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}), \quad i = 0, 1, \dots, \underline{N}$$

$$(6.22)$$

We note that ${}^{m}\underline{\alpha}^{i}|_{\underline{\alpha}}$; $i = 0, 1, ..., \underline{N}, \frac{\partial ({}^{m}\underline{\alpha}^{i})}{\partial ({}^{m}\underline{I}^{j})}|_{\underline{\alpha}}$; $j = 1, 2, ..., \underline{M}, \frac{\partial ({}^{m}\underline{\alpha}^{i})}{\partial \theta}|_{\underline{\alpha}}$; $i = 0, 1, ..., \underline{N}$ are functions of $\theta|_{\underline{\alpha}}$ and $({}^{m}\underline{I}^{j})_{\underline{\alpha}}$; $j = 1, 2, ..., \underline{M}$. We substitute (6.22) in (6.20) to obtain most general constitutive theory for [m]. Details are given

in the following

$$[m] = \left({}^{m}\underline{\alpha}^{0} \right|_{\underline{\Omega}} + \sum_{j=1}^{\underline{M}} \frac{\partial({}^{m}\underline{\alpha}^{0})}{\partial({}^{m}\underline{I}^{j})} \Big|_{\underline{\Omega}} \left({}^{m}\underline{I}^{j} - ({}^{m}\underline{I}^{j})_{\underline{\Omega}} \right) + \frac{\partial({}^{m}\underline{\alpha}^{0})}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) [I]$$

$$+ \sum_{i=1}^{\underline{N}} \left({}^{m}\underline{\alpha}^{i} \Big|_{\underline{\Omega}} + \sum_{j=1}^{\underline{M}} \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial({}^{m}\underline{I}^{j})} \Big|_{\underline{\Omega}} ({}^{m}\underline{I}^{j} - ({}^{m}\underline{I}^{j})_{\underline{\Omega}}) + \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) [{}^{m}\underline{G}^{i}]$$

$$(6.23)$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (6.23) that are defined in the current configuration and also grouping together those terms that are completely defined in the known configuration $\underline{\Omega}$. Let us define

$${}^{0}\underline{m}\big|_{\underline{\Omega}} = {}^{m}\underline{\alpha}^{0}\big|_{\underline{\Omega}} + \sum_{j=1}^{\underline{M}} \frac{\partial({}^{\sigma}\underline{\alpha}^{0})}{\partial({}^{\sigma}\underline{I}^{j})}\big|_{\underline{\Omega}} ({}^{m}\underline{I}^{j})_{\underline{\Omega}};$$

$${}^{m}\underline{a}_{j} = \frac{\partial({}^{m}\underline{\alpha}^{0})}{\partial({}^{m}\underline{I}^{j})}\big|_{\underline{\Omega}}, \ j = 1, 2, \dots, \underline{M}$$

$${}^{m}\underline{b}_{i} = {}^{m}\underline{\alpha}^{i}\big|_{\underline{\Omega}} - \sum_{j=1}^{\underline{M}} \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial({}^{m}\underline{I}^{j})}\big|_{\underline{\Omega}}; \ i = 1, 2, \dots, \underline{N}$$

$${}^{m}\underline{c}_{ij} = \frac{\partial({}^{m}\underline{\alpha}^{i})}{\partial({}^{m}\underline{I}^{j})}\big|_{\underline{\Omega}} \ i = 1, 2, \dots, \underline{N}. \ j = 1, 2, \dots, \underline{M}$$

$${}^{m}\underline{d}_{i} = -\frac{\partial({}^{m}\underline{\alpha}^{0})}{\partial\theta}\big|_{\underline{\Omega}}; \ i = 1, 2, \dots, \underline{N}$$

$${}^{m}\underline{\alpha}_{tm} = -\frac{\partial({}^{m}\underline{\alpha}^{0})}{\partial\theta}\big|_{\underline{\Omega}}$$

$$(6.24)$$

Using, (6.24) in (6.23) we can write (6.23) as

$$[m] = {}^{0} \underline{m} \Big|_{\underline{\Omega}} [I] + \sum_{j=1}^{\underline{M}} {}^{m} \underline{a}_{j} {}^{m} \underline{I}^{j} [I] + \sum_{i=1}^{\underline{N}} {}^{m} \underline{b}_{i} [{}^{m} \underline{G}^{i}] + \sum_{i=1}^{\underline{N}} \sum_{j=1}^{\underline{M}} {}^{m} \underline{c}_{ij} {}^{m} \underline{I}^{j} [{}^{m} \underline{G}^{i}] + \sum_{i=1}^{\underline{N}} {}^{m} \underline{d}_{i} (\theta - \theta_{\underline{\Omega}}) [{}^{m} \underline{G}^{i}] - {}^{m} \underline{\alpha}_{tm} (\theta - \theta_{\underline{\Omega}}) [I]$$

$$(6.25)$$

 ${}^{m}\underline{a}_{j}, {}^{m}\underline{b}_{i}, {}^{m}\underline{c}_{ij}, {}^{m}\underline{d}_{i}$ and ${}^{m}\underline{\alpha}_{tm}$ are material coefficients defined in known configuration $\underline{\Omega}$. The constitutive theory requires $(\underline{M} + \underline{N} + (\underline{N})(\underline{M}) + \underline{N} + 1)$ material coefficients. The material coefficients defined in (6.24) are functions of $\theta|_{\underline{\Omega}}$ and $({}^{m}\underline{f}^{j})|_{\underline{\Omega}}; j = 1, 2, \ldots, \underline{M}$. This constitutive theory is based on integrity (complete basis), hence is complete.

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6.3.1. Simplified linear rate constitutive theory of up to order ${}^{1}n$ for moment tensor \boldsymbol{m}

Using the general ordered rate constitutive theory of up to order ${}^{1}n$ in (6.25) we can consider simplified constitutive theories of up to order ${}^{1}n$ based on the following assumptions.

- (1) The constitutive theories are linear in the components of each argument tensor of [m].
- (2) We neglect all terms containing the products of the components of the argument tensors i.e the tensors containing the products of the generators and the invariants as well as the terms containing products of $(\theta \theta_{\underline{\alpha}})$ with generators and invariants.
- (3) We assume the constitutive theory for [m] to be independent of $\{g\}$.

Based on these assumptions, the constitutive theory for [m] will only contain tensor $\begin{bmatrix} \Theta \\ s \end{bmatrix}$, $\begin{bmatrix} \Theta \\ s \end{bmatrix}$; j = 1, 2, ..., n as generators and their traces as invariants, and we can write the following for [m]

$$[m] = {}^{0} \underline{m} \Big|_{\underline{\Omega}} [I] + \hat{a}_{1} [{}^{\Theta}_{s} D] + \hat{a}_{2} (\operatorname{tr} [{}^{\Theta}_{s} D]) [I] + \sum_{j=1}^{1} \hat{b}_{j}^{1} [{}^{\Theta}_{s} D_{[j]}] + \sum_{j=1}^{1} \hat{b}_{j}^{2} (\operatorname{tr} [{}^{\Theta}_{s} D_{[j]}]) [I] - ({}^{m} \underline{\alpha}_{tm}) (\theta - \theta_{\underline{\Omega}}) [I]$$

$$(6.26)$$

The material coefficients \hat{a}_1 , \hat{a}_2 , \hat{b}_j^1 , \hat{b}_j^2 are functions of $\theta_{\underline{\Omega}}$ and $({}^m \underline{I}{}^j)|_{\underline{\Omega}}$; $j = 1, 2, \ldots, \widetilde{M}$. We can write (6.26) in the following form using Voigt's notation (in the absence of first and the last terms in (6.26)).

$$\{m\} = [\hat{a}]\{{}^{\Theta}_{s}D\} + \sum_{j=1}^{1} [\hat{b}_{j}]\{{}^{\Theta}_{s}D_{[j]}\}$$
(6.27)

in which components of [m] in $\{m\}$ are arranged in the same manner as in (6.16) for $\{d(s\sigma)\}$ and $[\hat{a}]$ is obtained from $[\underline{a}]$ in (6.17) by replacing a_1 and a_2 with \hat{a}_1 and \hat{a}_2 and $[\hat{b}_j]$ by replacing a_1 and a_2 in (6.17) with \hat{b}_j^1 and \hat{b}_j^2 . In (6.27) $\{m\}$ is a linear function of the symmetric part of the rotation gradient tensor and its rates up to order 1n . We note that at the very least $[{}^{\Theta}_s D]$ and $[{}^{\Theta}_s D_{[1]}]$ are essential for energy storage and dissipation. $[{}^{\Theta}_s D_{[j]}]$; $j = 2, 3, \ldots, {}^1n$ incorporates higher order dissipation physics. Equation (6.27) provides many opportunities for a varied range of linear constitutive theories for [m].

Remarks

We note that $\operatorname{tr}[{}_{s}^{\Theta}D] = 0$ and also

$$tr[{}_{s}^{\Theta}D]_{[j]}] = 0; \ j = 1, 2, \dots, {}^{1}n$$
(6.28)

hence (6.25) and (6.26) and also (6.27) can be modified using (6.24). In case of (6.26) incorporating (6.28) will result in reduction of the number of invariants i.e \widetilde{M} will reduce. In (6.26) we can see its impact immediately. Using (6.28) in (6.26) we get

$$[m] = {}^{0}\underline{m}|_{\underline{\Omega}}[I] + \hat{a}_{1}[{}^{\Theta}_{s}D] + \sum_{j=1}^{{}^{1}n} \hat{b}_{j}^{1}[{}^{\Theta}_{s}D_{[j]}] - ({}^{m}\underline{\alpha}_{tm})(\theta - \theta_{\underline{\Omega}})[I]$$
(6.29)

Equations (6.27) remain same except in the definitions of $[\hat{a}]$ and $[\hat{b}_j]$, the coefficients \hat{a}_2 and $[\hat{b}_j^2]$; j = 1, 2, ..., n must be set to zero.

6.4. Ordered rate constitutive theories of orders n and ${}^{1}n$ for \boldsymbol{q} Recall the argument tensors of \boldsymbol{q} in (5.31)

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}, \, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, \dots, n, \, \overset{\boldsymbol{\Theta}}{\boldsymbol{s}} \boldsymbol{D}, \, \overset{\boldsymbol{\Theta}}{\boldsymbol{s}} \boldsymbol{D}_{[j]}; j = 1, 2, \dots, {}^{1}n, \, \boldsymbol{g}, \theta)$$
(6.30)

we must maintain these arguments of \boldsymbol{q} as there are no other conditions or restrictions that suggest eliminating any of them. Let $\{{}^{q}\!G^{i}\}; i = 1, 2, ..., \widetilde{N}$ be the combined generators of the argument tensors in (6.30) and ${}^{q}\!I^{j}; j = 1, 2, ..., \widetilde{M}$ \ldots, \widetilde{M} be combined invariants of the same argument tensors. Then we can express $\{q\}$ as a linear combination of $\{{}^{q}\!G^{i}\}; i = 1, 2, ..., \widetilde{N}$.

$$\{q\} = -\sum_{i=1}^{\tilde{N}} \tilde{\alpha}^{i} \{ {}^{q} \tilde{G}^{i} \}$$

$$(6.31)$$

The absence of unit vector in (6.31) is due to the fact that uniform temperature field does not contribute to $\{q\}$. The negative sign is due to the fact that a positive $\{q\}$ in the direction of the unit external vector on the surface of the volume of matter results in heat removal from the volume of matter. The coefficients ${}^{q}\!\alpha^{i}; i = 1, 2, \ldots, \widetilde{N}$ are function of θ and ${}^{q}\!\mathcal{I}^{j}; j = 1, 2, \ldots, \widetilde{M}$ in the current configuration. To determine material coefficients from ${}^{q}\!\alpha^{i}; i = 1, 2, \ldots, \widetilde{N}$ in (6.31), we consider Taylor series expansion of each ${}^{q}\!\alpha^{i}; i = 1, 2, \ldots, \widetilde{N}$ about a known configuration $\underline{\Omega}$ in θ and ${}^{q}\!\mathcal{I}^{j}; j = 1, 2, \ldots, \widetilde{M}$ and retain only up to linear terms in θ and the invariants.

$${}^{q}_{\underline{\alpha}^{i}} = {}^{q}_{\underline{\alpha}^{i}} \Big|_{\underline{\alpha}} + \sum_{j=1}^{\widetilde{M}} \frac{\partial ({}^{q}_{\underline{\alpha}^{i}})}{\partial ({}^{q}_{\underline{j}^{j}})} \Big|_{\underline{\alpha}} \left({}^{q}_{\underline{j}} - ({}^{q}_{\underline{j}^{j}})_{\underline{\alpha}} \right) + \frac{\partial ({}^{q}_{\underline{\alpha}^{i}})}{\partial \theta} \Big|_{\underline{\alpha}} (\theta - \theta_{\underline{\alpha}}), \quad i = 1, 2, \dots, \widetilde{N} \quad (6.32)$$

Here, ${}^{q}\underline{\alpha}^{i}|_{\underline{\alpha}}$; $\frac{\partial({}^{q}\underline{\alpha}^{i})}{\partial({}^{q}\underline{I}^{j})}|_{\underline{\alpha}}$; $j = 1, 2, ..., \widetilde{M}$ and $\frac{\partial({}^{q}\underline{\alpha}^{i})}{\partial\theta}|_{\underline{\alpha}}$; $i = 1, 2, ..., \widetilde{N}$ are functions of θ and ${}^{q}\underline{I}^{j}$; $j = 1, 2, ..., \widetilde{M}$ in the known configuration $\underline{\Omega}$ where as ${}^{q}\underline{\alpha}^{i}$ are functions of the same quantities but in the current configuration by substituting (6.32) in (6.31) we obtain the most general constitutive theory for $\{q\}$.

$$\{q\} = -\sum_{i=1}^{\widetilde{N}} \left({}^{q} \underline{\alpha}^{i} \Big|_{\underline{\Omega}} + \sum_{j=1}^{\widetilde{M}} \frac{\partial ({}^{q} \underline{\alpha}^{i})}{\partial ({}^{q} \underline{I}^{j})} \Big|_{\underline{\Omega}} \left({}^{q} \underline{I}^{j} - ({}^{q} \underline{I}^{j})_{\underline{\Omega}} \right) + \frac{\partial ({}^{q} \underline{\alpha}^{i})}{\partial \theta} \Big|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \{ {}^{q} \underline{G}^{i} \}$$
(6.33)

We collect coefficients (quantities defined in $\underline{\Omega}$) of the terms in (6.33) that are defined in the current configuration i.e coefficients of $\{{}^{q}\!\mathcal{G}^{i}\}, \, \underline{\mathcal{I}}^{j}\{{}^{q}\!\mathcal{G}^{i}\}$ and $(\theta - \theta_{\underline{\Omega}})\{{}^{q}\!\mathcal{G}^{i}\}.$

Let us define

$${}^{q}\underline{b}_{i} = {}^{q}\underline{\alpha}^{i}\big|_{\underline{\Omega}} - \sum_{j=1}^{\widetilde{M}} \frac{\partial({}^{q}\underline{\alpha}^{i})}{\partial(\overline{q}\underline{I}^{j})}\big|_{\underline{\Omega}}; i = 1, 2, \dots, \widetilde{N}$$
$${}^{q}\underline{c}_{ij} = \frac{\partial({}^{q}\underline{\alpha}^{i})}{\partial(\overline{q}\underline{I}^{j})}\big|_{\underline{\Omega}}; i = 1, 2, \dots, \widetilde{N}. \quad j = 1, 2, \dots, \widetilde{M}$$
$${}^{q}\underline{d}_{i} = \frac{\partial({}^{q}\underline{\alpha}^{i})}{\partial\theta}\Big|_{\underline{\Omega}}; i = 1, 2, \dots, \widetilde{N}$$
(6.34)

Using (6.34) in (6.31) we can write (6.31) as

$$\{q\} = -\sum_{i=1}^{\tilde{N}} {}^{q}\underline{b}_{i} \{{}^{q}\underline{G}^{i}\} - \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{M}} {}^{q}\underline{c}_{ij} {}^{q}\underline{I}^{j} \{{}^{q}\underline{G}^{i}\} - \sum_{i=1}^{\tilde{N}} {}^{q}\underline{d}_{i} (\theta - \theta_{\underline{\alpha}}) \{{}^{q}\underline{G}^{i}\}$$
(6.35)

In which ${}^{q}\underline{b}_{i}$, ${}^{q}\underline{c}_{ij}$ and ${}^{q}\underline{d}_{i}$ are material coefficients defined in known configuration $\underline{\Omega}$. This constitutive theory defined by (6.35) requires $(\widetilde{N} + \widetilde{N}\widetilde{M} + \widetilde{N})$ material coefficients which are functions of θ and ${}^{q}\underline{I}^{j}$; $j = 1, 2, \ldots, \widetilde{M}$ in the known configuration $\underline{\Omega}$. This constitutive theory for $\{q\}$ is based on integrity, hence is complete.

6.5. Ordered rate constitutive theories of order n = 1 and ${}^{1}n = 1$ for $[_{d}(_{s}\sigma)]$ and [m]

It is perhaps instructive to consider rate constitutive theories of order n = 1and ${}^{1}n = 1$ for deviatoric symmetric stress tensor $[d(s\sigma)]$ and moment tensor [m].

6.5.1. Constitutive theories of order n = 1 for $[d(s\sigma)]$

Following (6.14) and choosing n = 1, we obtain,

$$\begin{bmatrix} a(s\sigma) \end{bmatrix} = \underline{\sigma}^0 \Big|_{\underline{\alpha}} [I] + a_1[\varepsilon_{[0]}] + a_2(\operatorname{tr}([\varepsilon_{[0]}])[I] + b_1^1[\varepsilon_{[1]}] + b_1^2(\operatorname{tr}([\varepsilon_{[1]}])[I] - (\sigma \underline{\alpha}_{tm})(\theta - \theta_{\underline{\alpha}})[I] - (\sigma \underline{\alpha}_{tm})(\theta - \theta_{\underline{\alpha}})[I] \end{bmatrix}$$
(6.36)

Neglecting first and the last terms in (6.36) and using Voigt's notation we can write the following for (6.36) in matrix and vector notations.

$$\{d(s\sigma)\} = [\underline{a}]\{\varepsilon_{[0]}\} + [\underline{b}_1]\{\varepsilon_{[1]}\}$$

$$(6.37)$$

Matrix $[\underline{a}]$ is defined by (6.17). Matrix \underline{b}_1 has same structure as matrix $[\underline{a}]$ but a_1, a_2 in $[\underline{a}]$ are replaced by b_1^1 and b_1^2 . **Remarks**

- (1) This is the simplest possible constitutive theory for $[_d(_s\sigma)]$ for incompressible thermoviscoelastic solids.
- (2) This constitutive theory requires two material coefficients a_1 , a_2 (Lame's constant) or modulus of elasticity E and Poisson's ratio ν as a_1 and a_2 can be expressed in terms of E and ν and vice versa.
- (3) This constitutive theory also requires two material coefficients b_1^1 and b_1^2 relating strain rates to $[d(s\sigma)]$. This is due to the fact that an elongation rate in a direction must be accompanied by a corresponding rate in the directions that are orthogonal.
- (4) $\{\varepsilon_{[0]}\}\$ and $\{\varepsilon_{[1]}\}\$ in (6.37) result in storage of some mechanical work (strain energy) and dissipation of some mechanical energy respectively.

6.5.2. Constitutive theories of order ${}^{1}n = 1$ for [m]

Following (6.26) and choosing ${}^{1}n = 1$, we obtain

$$[m] = {}^{0} \underline{m} \Big|_{\underline{\alpha}} [I] + \hat{a}_{1} [{}^{\Theta}_{s} D] + \hat{a}_{2} (\operatorname{tr} [{}^{\Theta}_{s} D]) [I] + \hat{b}_{1}^{1} [{}^{\Theta}_{s} D_{[1]}] + \hat{b}_{1}^{2} (\operatorname{tr} [{}^{\Theta}_{s} D_{[1]}]) [I] - ({}^{m} \underline{\alpha}_{tm}) (\theta - \theta_{\underline{\alpha}}) [I]$$

$$(6.38)$$

First, we note that

$$\operatorname{tr}[{}_{s}^{\Theta}D] = 0 \qquad \operatorname{tr}[{}_{s}^{\Theta}D_{[1]}] = 0 \tag{6.39}$$

using (6.39) in (6.38), neglecting first and the last terms in (6.38), (6.38) in Voigt's matrix and vector notations can be written as

$$\{m\} = [\hat{a}]\{{}^{\Theta}_{s}D\} + [\hat{b}_{1}]\{{}^{\Theta}_{s}D_{[1]}\}$$
(6.40)

In (6.40), $\{{}_{s}^{\Theta}D\}$ results in additional energy storage (strain energy) due to mechanical work and $\{{}_{s}^{\Theta}D_{[1]}\}$ causes additional dissipation due to mechanical work. The constitutive theory for [m] in (6.40) requires only two material coefficients, one associated with additional energy storage and the other resulting in additional dissipation.

6.6. Simplified constitutive theories for q

Much simpler (but with limitations) constitutive theories for $\{q\}$ can be derived if we limit its argument tensors. Consider a constitutive theory for $\{q\}$ using $\{g\}$ and θ as its argument tensors.

$$\{q\} = \{q\}(\{g\}, \theta) \tag{6.41}$$

In this case we only have one generator and one invariant

$$\{{}^{q}G^{1}\} = \{g\} \text{ and } \{{}^{q}I^{1}\} = \{g\}^{T}\{g\}; \qquad \widetilde{N} = 1, \, \widetilde{M} = 1$$
 (6.42)

Following the general derivations in section 6.4 we can write the following when (6.42) in considered.

$$\{q\} = -{}^{q}\underline{b}_{1}\{g\} - {}^{q}\underline{c}_{11}(\{g\}^{T}\{g\})\{g\} - {}^{q}\underline{d}_{1}(\theta - \theta_{\underline{\alpha}})\{g\}$$
(6.43)

in which the material coefficients ${}^{q}\underline{b}_{1}$, ${}^{q}\underline{c}_{11}$ and ${}^{q}\underline{d}_{1}$ are defined in (6.34).

This constitutive theory is cubic in $\{g\}$, requires only three material coefficients and is the most general complete constitutive theory based on (6.41). If we denote ${}^{q}\underline{b}_{1} = k_{1}|_{\underline{\rho}}$ and ${}^{q}\underline{c}_{11} = k_{2}|_{\underline{\rho}}$ then (6.43) can be written as

$$\{q\} = -k_1 \Big|_{\underline{\alpha}} \{g\} - k_2 \Big|_{\underline{\alpha}} (\{g\}^T \{g\}) \{g\} - {}^q \underline{d}_1 (\theta - \theta_{\underline{\alpha}}) \{g\}$$
(6.44)

If we neglect the last term in (6.44) (as commonly done in published works) we obtain

$$\{q\} = -k_1 \Big|_{\underline{\alpha}} \{g\} - k_2 \Big|_{\underline{\alpha}} (\{g\}^T \{g\}) \{g\}$$
(6.45)

If we assume $\{q\}$ is a linear function of $\{g\}$, then (6.45) reduces to

$$\{q\} = -k_1 \Big|_{\underline{\alpha}} \{g\} \tag{6.46}$$

Equation (6.46) is Fourier heat conduction law in which the thermal conductivity $k_1|_{\underline{\Omega}}$ can be a function of $\theta|_{\underline{\Omega}}$ and $\{g\}^T\{g\}$. We can also derive constitutive theory (6.46) using $\{q\}^T\{g\} \leq 0$ resulting from the entropy inequality. This derivation is standard and can be found in reference [6]

7. Summary and conclusion

In this paper ordered rate constitutive theories of orders n and n for homogeneous, isotropic and incompressible internal polar non-classical thermoviscoelastic solids without memory are presented. These solids contain elastic as well as dissipative mechanisms, hence the constitutive theories for such solids can not be derived using the same approaches that are used for thermoelastic solids such as those based on strain energy density function. The dependent variables in the constitutive theories are established by examining the conservation and balance laws in conjunction with principle of casualty [6, 8, 9]. By examining the conditions resulting from entropy inequality and after introducing decomposition of the stress tensor into symmetric and skew symmetric tensors we finally arrive at Φ , η , \boldsymbol{q} , ${}_{s}\boldsymbol{\sigma}$ and \boldsymbol{m} as dependent variables in the constitutive theories. Conjugate pairs in entropy inequality and the physics of energy storage and dissipation due to ${}^{s}_{d}\sigma$ and \boldsymbol{m} suggest that at least $\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]}; i = 1, 2, ..., n$ and $\boldsymbol{\theta}$ must be the argument tensors of $_{s}\boldsymbol{\sigma}$, likewise ${}^{\Theta}_{s}\boldsymbol{D}_{[0]}, {}^{\Theta}_{s}\boldsymbol{D}_{[j]}; j = 1, 2, \ldots, {}^{1}n \text{ and } \theta \text{ must at least be the argument tensors of}$ **m**. Temperature gradient \boldsymbol{g} and $\boldsymbol{\theta}$ are essential argument tensors of \boldsymbol{q} . Based on principle of equipresence [6, 8, 9] we choose totality of the argument tensors discussed above as argument tensors of each of the dependent variables $\Phi, \eta, q, s\sigma$ and **m** in the constitutive theories. Using Φ and its argument tensors we can determine $\dot{\Phi}$. When $\dot{\Phi}$ is substituted in the entropy inequality we obtain equalities and inequalities that must hold in order to satisfy entropy inequality. These conditions help in reducing the number of dependent variables $(\eta \text{ is eliminated})$ in the constitutive theories as well as help in reducing the number of argument tensors for some of the remaining dependent variables. The conditions resulting from the entropy inequality require further decomposition of ${}_{s}\boldsymbol{\sigma}$ into $({}_{e}({}_{s}\boldsymbol{\sigma}))$ and $({}_{d}({}_{s}\boldsymbol{\sigma}))$ stress tensors and likewise decomposition of **m** into $_{e}\boldsymbol{m}$ and $_{d}\boldsymbol{m}$. It is shown that $_{e}\boldsymbol{m}$ does not result in rate of work, hence we can use \boldsymbol{m} or $d\boldsymbol{m}$ in the constitutive theories. In the derivation of the constitutive theory for the deviatoric part of the symmetric stress tensor, strain tensor, strain rate tensors up to orders n, temperature gradient tensor g and temperature θ are its argument tensors. The symmetric part of the rotation gradient tensor, its rates up to order ${}^{1}n$, temperature gradient tensor g and temperature θ are argument tensors of the moment tensor m (or the deviatoric moment tensor) in the derivations of the constitutive theory for the moment tensor. A most general form of the constitutive theory is presented for heat vector in which its argument tensors are strains, strain rates up to orders n, symmetric part of the rotation gradient tensor, its rates up to orders n, temperature gradient tensor g and temperature θ . The second law of thermodynamics in conjunction with theory of generators and invariants forms the basis for the derivations of the constitutive theories. We make some specific remarks pertaining to the constitutive theories presented in this paper.

- (1) In every deforming solid continua the Jacobian of deformation or displacement gradient tensor varies between neighboring material parts. Its decomposition into pure stretch and rotation suggest that varying rotations between material point if resisted by the solid matter will result in conjugate moment tensor. This physics exists in all homogeneous, isotropic deforming solids but is not considered in the classical continuum theories used presently for homogeneous, isotropic, incompressible solid continua. This new physics due to rotations and moment tensors results in additional storage of strain energy as well additional dissipation. This deformation physics of solid continua with elasticity and dissipation is described by the "internal polar non-classical" continuum theories presented by Surana et. al [1,2].
- (2) Some energy storage is due to ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}_{[0]}$ while the corresponding dissipation mechanism is due to ${}_{d}({}_{s}\boldsymbol{\sigma})$ and $\boldsymbol{\varepsilon}_{[i]}; i = 1, 2, ..., n$ as in classical ordered rate theories for classical thermoviscoelastic solids without memory [26].
- (3) The additional energy storage is due to \boldsymbol{m} and ${}_{s}^{\Theta}\boldsymbol{D}_{[0]}$ while the corresponding additional dissipation mechanism is due to \boldsymbol{m} and ${}_{s}^{\Theta}\boldsymbol{D}_{[j]}; j = 1, 2, ..., {}^{1}n$.
- (4) It is shown that the constitutive theory for heat vector \boldsymbol{q} is rather complex as it requires considerations of all argument tensors due to the fact that

there is no justifiable basis for reducing them. Simplified theories for \boldsymbol{q} can be derived based on our choice of argument tensors or using some rationale for eliminating some from the complete list of argument tensors.

- (5) These internal polar non-classical theories and associated constitutive theories point out that homogeneous isotropic thermoviscoelastic solids without memory offer more resistance during deformation and contain more dissipation than what is currently present in the constitutive theories based on classical continuum theories.
- (6) Choice of n and ¹n in the constitutive theories for _d(_sσ) and *m* limits the dissipation mechanism due to _d(_sσ) and *m*. At the very least ε_[0], ε_[1] *i.e.* n = 1 and ^Θ_sD_[0], ^Θ_sD_[1] ¹n = 1 are essential for energy storage and dissipation. Choice of n = 1 and ¹n = 1 will provide the simplest possible constitutive theories for _d(_sσ) and *m* with the least number of material coefficients.
- (7) The constitutive theories derived using combined generators and invariants of the argument tensors are based on integrity, hence are complete i.e. consider complete basis that is determined using the argument tensors.
- (8) We note that in the present work determination of material coefficients from the coefficients used in the linear combination of the combined generators to express the deviatoric part of the symmetric stress tensor, heat vector, and moment tensor requires use of Taylor series about a known configuration. This automatically forces the determination of the material coefficients in a known configuration and not in the current configuration. In all presently used works, this is not the case. The use of material coefficients in the current configurations is justified when the current configurations are in close proximity to each other, but this cannot be supported by the derivations presented in this work.
- (9) The work presented in this paper is a most general framework for deriving ordered rate constitutive theories of order n and ${}^{1}n$ for thermoviscoelastic solids without memory that are homogeneous, isotropic and incompressible solids and are described by conservation and balance laws derived using internal polar non-classical continuum theories. The general constitutive

theories presented in this paper easily permit many simplified theories by limiting argument tensors and by further simplification after Taylor series expansion of the coefficients in the linear combination. Solutions of model problems and numerical studies using the constitutive theories presented here will be considered in a follow up paper.

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